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PROFILE – OPTIMISE – COMPLY

Equivalent Risky Allocation

The new ERA of risk measurement for heterogeneous investors

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Modern Portfolio Theory introduced by Markowitz [1952] defines risk as the variance of portfolio return. Since its publication, many studies have shown that higher moment of the distribution of returns are relevant for asset allocation decisions (Samuelson [1970], Kraus and Litzenberger [1976], Friend and Westerfield [1980] among others). Although several risk measures taking into account higher moments of distribution have been proposed in the literature (see for instance Coombs and Lehner [1981] or Favre and Galeano [2002]), they do not take into account the investor's "perception" of financial risks. We define this perception as the subjective judgment of an investor over the characteristics and severity of a potential loss. Undeniably, investors display heterogeneous attitudes towards the notion of "risk". We can identify two extreme behaviors. Some investors put very strong emphasis on the stability of returns around their mean and put significantly less weight on extreme but rare losses. These are close to traditional "mean-variance investors", driven by the variance of asset returns in the scope of the Modern Portfolio Theory framework. At the other end of the spectrum, some investors primarily care about tail risk. They are sensitive to the threat of a shortfall with respect to a threshold level of wealth. These investors fall closer to the "mean-VaR investors" such as in the framework of Favre and Galeano [2002].

The objectives of this paper are twofold. We first aim to propose a risk measure derived from the Expected Utility Theory that explicitly takes into account the risk perception of the investor. This measure is applicable to various kinds of investors, from one extreme to another. It is derived from Bell [1988]'s linear plus exponential (linex) utility function, characterized by two parameters: one for the risk aversion, and one for the risk perception. The design of our risk measure, called Equivalent Risky Allocation (ERA), makes it practical, easily interpretable and comparable from one definition of risk to another. The ERA of a portfolio is the percentage invested in a specified benchmark (the rest being invested in the risk-free asset) that delivers the same risk as this portfolio. This ERA has the advantage of providing a single, homogeneous value for each portfolio, while properly accounting for the heterogeneity of the investors[1]. We demonstrate that for a same value of ERA, there exist several optimal portfolio allocations depending on the risk perception of the investor.

Our second objective is to examine, with this new risk measure, the relevance of explicitly accounting for investors' risk perceptions in their portfolio allocation decisions, both static and dynamic. Considering several investors differing by their risk profile (risk perception and risk aversion) but confronted with the same set of asset classes and allocation constraints, we observe that, under a passive buy-and-hold strategy, the ERA obtained with Bell's Risk Measure allows a better control of actual risk

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[1] This is not the case with the generalized Sharpe ratio of Zakamouline and Koekebakker [2009] who propose a performance measure including higher moments of distribution derived from the hyperbolic absolute risk aversion (HARA) utility function. This measure requires the assumption that all investors share the same perception of risk.

exposures than with the variance, that only addresses the stability of returns.

Under an active strategy, reallocating our portfolios every four weeks, applying the ERA with the proper risk perception enables the portfolio manager to tightly monitor and control its risk exposure. Portfolio risk, measured four weeks after the allocation, moves never further than 1% from its target. We also illustrate the impact of a different perception of risk on the allocations of the optimal portfolios. Finally, we show the impact of a wrong profiling through both an in-sample and out-of-sample check of the portfolio ERA measures with another risk profile than the one used for portfolio allocations. This acid test emphasizes the strong consistency of Bell [1988]'s risk measure and the quality of the portfolios composed by using this characterization of investor preferences.

The risk aversion - risk perception approach

Bell's risk measure

Bell [1988] has proven that the only utility function compatible with a decision maker that prefers more money to less, wishes to obey the axioms of expected utility, is decreasingly risk averse at all wealth levels, wishes to obey the one-switch rule[2] and will approach risk neutrality for small gambles when extremely rich, is the linex utility function:

$$U(W) = W - be^{-cW} \qquad (1)$$

where W is the wealth level, b the risk aversion coefficient and c the risk perception coefficient. Both coefficients are positive and investor-specific.

Financial decision making under uncertainty is an issue of trading off risk against return. For that purpose, Bell [1995] derives from his utility function an independent risk measure. His point is that the expected value of his utility function is a function of a measure of return, $\pi(\tilde{x})$, a measure of risk, $R(\tilde{x})$ and of the initial wealth, W0. Writing the evaluation of an alternative as follows:

$$E[U(W_0 + \tilde{x})\,] = W_0 + \pi(\tilde{x}) - be^{-cW_0}\, E[e^{-c\tilde{x}}\,] (2)$$

The only definition of risk compatible with Bell's assumptions is the following:

$$R(\tilde{x}) \;=\; E[e^{-c(\tilde{x}-\pi(\tilde{x}))}\,]. \qquad (3)$$

Because of parameter c which varies from one person to another, this measure is not unique but specific to each investor. The main advantage of this risk measure is that it includes, as special cases, many other measures of risk previously proposed in the literature.

Hlawitschka [1994] demonstrates that the best quality of the approximation of Bell's risk measure is achieved with the first fourth-order moments of the distribution. Consider the initial wealth of the investor W0 and the global amount invested in the risky asset I. The risk premium on this amount is equal to x = θ – r, where θ is the return of the risky asset, and r is the risk-free return. The expected utility becomes:

$$E\big[U\big(W(x)\big)\big] = W_0(1+r) + I\bar{x}$$
$$-b\, e^{-c(W_0(1+r)+Ix)} E\big[e^{-cIx(x-\bar{x})}\big] \qquad (5)$$

The Taylor series expansion of this expression around the mean is then:

$$E[U(W)] = W_0(1+r) + I\bar{x}$$
$$-be^{-cW_0(1+r)}e^{-cI\bar{x}}\Big(1 + \tfrac{1}{2}c^2I^2V_x - \tfrac{1}{6}c^3I^3S_x + \tfrac{1}{24}c^4I^4K_x\Big) \qquad (6)$$

Where

$V_x = E[\theta - \tilde{\theta}\,]^2$, $S_x = E\big[\theta - \tilde{\theta}\,\big]^3$ and $K_x = E[\theta - \tilde{\theta}\,]^4$.

From this equation, we get the risk measure R_x, for a standard unit of wealth:

$$R_x = \tfrac{1}{2}V_x - \tfrac{1}{6}CS_x + \tfrac{1}{24}C^2V_x, \qquad (7)$$

where $C \equiv c\dfrac{I}{W_0}$ is the product of the (intrinsic) risk perception coefficient, c, and the proportion of risky investment in the total wealth.

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[2] "A decision maker obeys the *one-switch rule* if, for every pair of alternatives whose ranking is not independent of wealth level, there exists a wealth level above which one alternative is preferred, below which the other is preferred." (Bell [1988]).

An example is helpful in order to better understand the role of the risk perception parameter C. Suppose three different investors, respectively characterized by C equal to 5, 17 and 34.[3] Each one is asked to choose between two assets characterized by the same expected return and the same variance (equal to 0.0029), but with different skewness and kurtosis. Exhibit 1 provides the coefficient applied to the variance, skewness and kurtosis and their respective weights (between brackets) given by each investor. We can see from Exhibit 1 that as C increases, more weight is put on the skewness and kurtosis.

Exhibit 1

Illustration of the weight given to the 2nd, 3rd and 4th moment of the distribution in Bell Risk Measure for 3 different value of C, the risk perception parameter.

Perception parameter, C	Coefficients multiplying :		
	Variance	Skewness	Kurtosis
5	0.50 (21%)	-0.83 (35%)	1.04 (44%)
17	0.50 (3%)	-2.83 (18%)	12.04 (78%)
34	0.50 (1%)	-5.67 (10%)	48.17 (88%)

Comparing two assets with the same expected return and the same variance, but with different skewness and kurtosis, we can see the evolution of the risk measures (see Exhibit 2). Whereas an investor using a traditional mean-variance approach will exhibit prefect indifference between both assets, the investor using the Bell Risk Measure will be more and more inclined to select Asset Y as her coefficient C gets bigger.

Exhibit 2

Illustration of the Bell Risk Measure's value for two assets characterized by the same return and variance, but different skewness and kurtosis.

Asset	Vx	Sx	Kx	Bell Risk Measure		
				C = 5	C =17	C =34
A	0.0029	-0.0002	0.0001	0.0016	0.0025	0.0047
B	0.0029	0.0001	0.0000	0.0014	0.0012	0.0009

Equivalent risky allocation

In order to make the Bell Risk Measure easily interpretable and comparable, we propose to express this measure in terms of an Equivalent Risky Allocation (ERA). The ERA of a portfolio is the percentage invested in a specified benchmark (the rest being invested in the risk-free asset) that delivers the same risk as this portfolio. The benchmark can be any portfolio used as a reference, and is not constrained to be ex-ante or ex-post efficient. This makes our measure very simple and practical. In the mean-variance context, the ERA can be compared to a generalization of the weight put in a benchmark portfolio along the Capital Allocation Line[4].

As the risk perception differs from one investor to another, there will be as many ERAs associated to a portfolio as the number of different investors. This measure provides a single index value for each portfolio, while properly accounting for the heterogeneity of the investors. The ERA can be applied to any measure of risk, as long as one uses the same measure to measure the risk of the studied asset and the risk of the benchmark. If the risk measure is homogeneous of degree 1 with the weight in the risky asset, such as the variance and the Bell Risk Measure, the formula of the ERA is:

$$\mathrm{ERA}_x = \frac{\mathrm{R}_x}{\mathrm{R}_\mathrm{B}} \qquad (8)$$

where R_B is the risk value of the selected benchmark.

To illustrate this measure, consider the Asset X in the previous example. Take S&P500 as benchmark, and the variance as the risk measure. The monthly variance of the S&P500 is equal to 0.0023. The ERA is then:

ERA_A= 0.0029 / 0.0023 = 126%

meaning that asset X is 26% riskier than the benchmark in terms of volatility, or that one needs to invest 126% of her wealth in the S&P500 (and borrow the 26% at the risk-free

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[3] As Bell does not propose any value for *C*, we derive its highest value empirically, by constructing optimal portfolios on the basis of four assets: a risk-free one, a low-variance-high kurtosis one, a high-variance-low kurtosis one and a high-variance-high kurtosis one. After setting the weights of the risk-free and the high-risk assets equal, the *C* producing the portfolio with the highest weight in the high-variance was equal to 34, and the *C* producing the portfolio with the highest weight in the high-kurtosis asset was equal to 5.

[4] It is not necessary to define the benchmark as the market portfolio. But if it is, the Capital Allocation Line is the Capital Market Line.

rate) in order to obtain the same risk level (here measured by variance) than her asset X.

With this approach, we are henceforth able to measure the risk of any portfolio with a single metric, irrespective of how the investor perceives the notion of risk. This is helpful in order to characterize the evolution of portfolios allocated with different risk-return optimization rules, as we do in the empirical section.

Data and Methodology

To compute optimal portfolios, we consider the weekly returns of nine equity indices and one bond index, for the period of January 7th, 2000 to August 14th, 2009. The equity indices are S&P 500, S&P 500 Growth, S&P 500 Value, S&P 400, S&P 400 Growth, S&P 400 Value, S&P 600, S&P 600 Growth and S&P 600 Value total return indices. The bond index is JPM US Aggregate bond index total return.

Exhibit 3 displays the mean, minimum and maximum weekly returns of the 10 assets, over the whole period (502 weeks), as well as their standard deviation, skewness and kurtosis. The sample means of the series do not really reflect the return we can expect for their respective indices. The equity indices, riskier than the bond index, display means lower or close from the bond index and are even negative for S&P500 and S&P500 Growth.

Exhibit 3

Descriptive statistics of the weekly returns' distributions of the indices.

	Mean (*10^{-3})	Min	Max	Stand. Dev	Stand. Sx	Stand. Kx
S&P500	-0.4	-0.20	0.11	0.03	-0.85	7.14
S&P500 Growth	-0.4	-0.21	0.14	0.04	-0.67	4.44
S&P500 Value	1.0	-0.24	0.19	0.04	-0.62	7.63
S&P400	1.0	-0.19	0.15	0.03	-0.65	5.50
S&P400 Growth	1.1	-0.18	0.14	0.04	-0.46	3.31
S&P400 Value	1.6	-0.19	0.20	0.04	-0.55	7.81
S&P600	1.0	-0.16	0.14	0.03	-0.57	3.59
S&P600 Growth	1.4	-0.17	0.14	0.04	-0.58	3.54
S&P600 Value	1.9	-0.21	0.20	0.04	-0.41	5.46
US Corp. Bond	1.2	-0.02	0.02	0.01	-0.50	1.01

As we are first interested in the risk profile of the assets, we adjust the returns of the series using the Black and Litterman [1992] model, smoothing the expected return through a market equilibrium model and combining our own view of the market returns with the equilibrium model's returns. The main inputs of the Black-Litterman model are the "average" risk aversion, set to 2.5 in He and Litterman [2002], the variance-covariance matrix of the returns[5], and the views we have of the market returns.

In order to specify our view, we go along with the Fama and French [1992] study, showing that, on average, Growth companies have lower returns than Value companies, and that small companies have higher returns than larger ones. Following the same methodology on the period from January 2000 to September 2008 (we excluded financial crisis data), we found an additional 0.23% weekly return for Value firms and of 0.13% for small companies. Moreover, given the high risk profiles compared to the bond index, we add the view that the bond index return must exceed the LIBOR 1M by its average spread for the last 18 months.

Exhibit 4 displays our view of the returns of the equity indices. As required by the Black-Litterman model, the sum of the weights given to the assets in a view is equal to 0 (resp. 1) when the view is relative (resp. absolute). The last line represents the vector of the adjustment to apply to the selected assets' returns, expressed in weekly returns.

Exhibit 4

Inputs of the Black-Litterman model: implicated assets, estimated returns and level of confidence of our views.

	View 1	View 2	View 3	View 4	View 5
S&P500	- 1/3	0	0	-1/3	0
S&P500 Growth	- 1/3	0	-1/3	0	0
S&P500 Value	- 1/3	0	1/3	1/3	0
S&P 400	0	- 1/3	0	-1/3	0
S&P 400 Growth	0	- 1/3	-1/3	0	0
S&P400 Value	0	- 1/3	1/3	1/3	0
S&P 600	1/3	1/3	0	-1/3	0
S&P 600 Growth	1/3	1/3	-1/3	0	0
S&P600 Value	1/3	1/3	1/3	1/3	0
US Corp. Bond	0	0	0	0	1
Estimated returns	0.13%	0.07%	0.23%	0.12%	0.09%

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[5] We use an estimation window of 18 months.

Exhibit 5 shows the evolution of the posterior expected returns of the indices. We apply the model over a rolling window of 78 weeks for the whole period. The absolute value of the bond index presented in the table is its average value for the whole period.

Exhibit 5

Average expected returns of the equity indices, before, during and after the implementation of the Black-Litterman approach.

	Historical	Equilibrium		View Adjusted	
		Mean	Std dev.	Mean	Std. dev.
S&P500	-0.04%	0.15%	0.13%	0.10%	0.38%
S&P500 Growth	-0.04%	0.19%	0.15%	-0.11%	0.55%
S&P500 Value	0.10%	0.19%	0.21%	0.56%	0.46%
S&P 600	0.10%	0.18%	0.15%	0.22%	0.42%
S&P 600 Growth	0.11%	0.20%	0.15%	0.02%	0.50%
S&P600 Value	0.16%	0.18%	0.20%	0.59%	0.46%
S&P 400	0.10%	0.19%	0.15%	0.42%	0.43%
S&P 400 Growth	0.14%	0.20%	0.16%	0.33%	0.45%
S&P400 Value	0.19%	0.20%	0.21%	0.85%	0.52%
US Corp. Bond	0.12%	-0.01%	0.01%	0.06%	0.02%

Once the expected returns are smoothened with the Black-Litterman model, we construct defensive, average and aggressive optimal portfolios, that is, with a maximum ERA of respectively 50%, 75% and 100%. The defensive (resp. average, aggressive) portfolio is therefore constructed such as its risk is half (resp. 75%, equal to) the risk of an equally weighted portfolio of the 9 equity indices (our benchmark).

We build these portfolios for four types of investor's profiles:

- A "Markowitz" investor, for whom we construct an optimal portfolio maximizing her expected return for a given variance;
- A protective investor, more affected by extreme losses than variability. Her portfolio is constructed using the Bell risk measure with a high C, equal to 34;
- A median investor, characterized by a medium C, equal to 17;
- A stable investor, more affected by variability than extreme events, characterized by a low C, equal to 5.

Results

In order to compare our twelve optimal portfolios computed with 4 different risk measures for three levels of risk, the risk values are systematically expressed in ERA. In the first sub-section we test the time consistency of the Bell Measure for a fixed-weights portfolio, by observing the evolution of the ERA over time. Then, we test the time consistency of the measure for a rebalanced portfolio. Finally, we test the portfolio consistency by recomputing the risk measures for the rebuilt historical of the rebalanced portfolios. Note that all portfolios are constructed under the constraint that the maximum weight of each equity index is set to 20%.

Time consistency with no rebalancing

To test the coherence of the risk measure over time, we construct 12 optimal portfolios, i.e. for 3 levels of risk and for 4 different perceptions of risk, every 4 weeks, starting on June 29th, 2001, until July 24th, 2009, and we observe the evolution of their ERA.

Exhibit 6 presents the average optimal allocations for the 12 optimal portfolios and compares their ERA on the day of the optimization for the 4 investors. The ERA is highlighted when the right risk measure is used for the specified investor.

Using the wrong risk measure for the investors can make them bear a too high risk level or miss a higher expected return by reducing too much the risk taken. Indeed, for defensive and average-risk portfolios, if we use a Bell Risk Measure with a high C (17 or 34), the optimal portfolios are too risky for the stable and Markowitz investors, as their ERA are higher than the required value (50% and 75%). On the other hand, a protective investor, for whom we have computed a defensive optimal portfolio using Markowitz, has a portfolio with too little risk, with a potential loss in expected return.

Exhibit 6

Average Equivalent Risky Allocation of optimal portfolios for 4 different investors and average allocations of these portfolios.

ERA	Risk Measure	Investors				Weights										
		Protective	Median	Stable	Markowitz	S&P 500	S&P 500 Growth	S&P 500 Value	S&P 400	S&P 400 Growth	S&P 400 Value	S&P 600	S&P 600 Growth	S&P 600 Value	Corp Bond	Average Std dev
Defensive (ERA = 50%)	Var	48%	50%	50%	**50%**			0.176			0.200	0.107	0.018	0.200	0.299	2.32%
	Bell $_{C=5}$	46%	49%	**50%**	50%			0.177			0.198	0.106	0.020	0.200	0.299	2.49%
	Bell $_{C=17}$	47%	**50%**	52%	52%			0.178			0.199	0.110	0.025	0.200	0.288	2.48%
	Bell $_{C=34}$	**50%**	53%	54%	55%			0.179			0.199	0.122	0.030	0.200	0.270	2.51%
Average (ERA = 75%)	Var	77%	76%	75%	**75%**	0.002		0.185	0.001		0.198	0.195	0.065	0.200	0.154	2.21%
	Bell $_{C=5}$	73%	74%	**75%**	75%	0.002		0.185	0.001		0.198	0.194	0.066	0.200	0.154	2.22%
	Bell $_{C=17}$	74%	**75%**	76%	76%	0.003		0.184	0.001		0.197	0.195	0.070	0.200	0.150	2.28%
	Bell $_{C=34}$	**75%**	76%	77%	77%	0.001		0.186	0.005		0.195	0.195	0.073	0.200	0.144	2.32%
Aggressive (ERA = 100%)	Var	108%	104%	101%	**100%**	0.012	0.042	0.173	0.007		0.174	0.197	0.147	0.200	0.047	3.71%
	Bell $_{C=5}$	101%	101%	**100%**	100%	0.010	0.036	0.174	0.007	0.001	0.180	0.197	0.150	0.200	0.046	3.42%
	Bell $_{C=17}$	101%	**100%**	99%	99%	0.009	0.013	0.185	0.005	0.003	0.189	0.200	0.151	0.200	0.045	2.70%
	Bell $_{C=34}$	**100%**	99%	98%	98%	0.003		0.187	0.009	0.008	0.195	0.200	0.150	0.200	0.048	2.22%

The portfolios are optimized every 4 weeks on the basis of the preceding 78 weekly returns (from June, 29th, 2001 to July, 24th, 2009) of 10 indices, with 4 different risk measures. The indices are S&P500, S&P500 Growth, S&P500 Value, S&P400, S&P400 Growth, S&P400 Value, S&P600, S&P600 Growth, S&P600 Value and JPM US Corp Bond index. The risk measures are the variance, the Bell's Risk Measure (Bell) with a perception parameter C equal to 5, equal to 17 and equal to 34. Under each risk measure, we construct three portfolios: a defensive (ERA = 50%), an average-risk (ERA=75%) and a dynamic (ERA = 100%) portfolio. The ERA is the value of the risk measure of the optimal portfolio, divided by the value of the same risk measure for the benchmark. The benchmark is an equally weighted portfolio of the 9 equity indices.

The observation of these average values already shows us the complexity induced by the various notions of risk held by the investors. Indeed, a protective investor is not always the one with the highest allocation in bonds. Taking a look at the average allocations of the portfolios (right part of Exhibit 6), the main difference between the defensive portfolios is between the allocation in the S&P600, S&P600 Growth and the bond index. For a low level of risk, a protective investor will only invest, on average, 27% in the bond index, whereas the Markowitz investor will have, on average, 30%, invested in the bond index.

For the average risk portfolios allocations, the differences between investors are marginal, leading to smaller variations for the value of the ERAs when a wrong profiling is selected. On the opposite, the aggressive portfolios differ more significantly among investors, with more volatile allocations (last column). The differences in the allocations most affect the S&P500 Growth and Value, and the S&P400 Growth and Value.

We then analyze the evolution of the ERA of each individual portfolio over time. Each portfolio's ERA is computed every week after the optimization until 3 years (156 weeks) later. The procedure is reproduced over a rolling window of 18 months, so that we get 78 observations for each week. Exhibit 7 displays, for the medium-risk portfolios[6], the evolution of the averages, the 5% and 95% quantiles of these ERAs, according to the number of weeks elapsed since the allocation date. The dashed lines represent the target value of the portfolios optimizations (75% in this example)[7]. The value taken by the ERAs after three years is on average equal to respectively 86%, 88%, 90% and 92% for resp. the Bell$_{C=34}$, Bell$_{C=17}$, Bell$_{C=5}$ and variance, which means that the portfolios optimized with the traditional variance approach suffer the most from this increase in the bond risk.

The values of the 95% quantile show the maximum value the ERA can take in 95% of the cases. It takes its highest value under the Markowitz approach.

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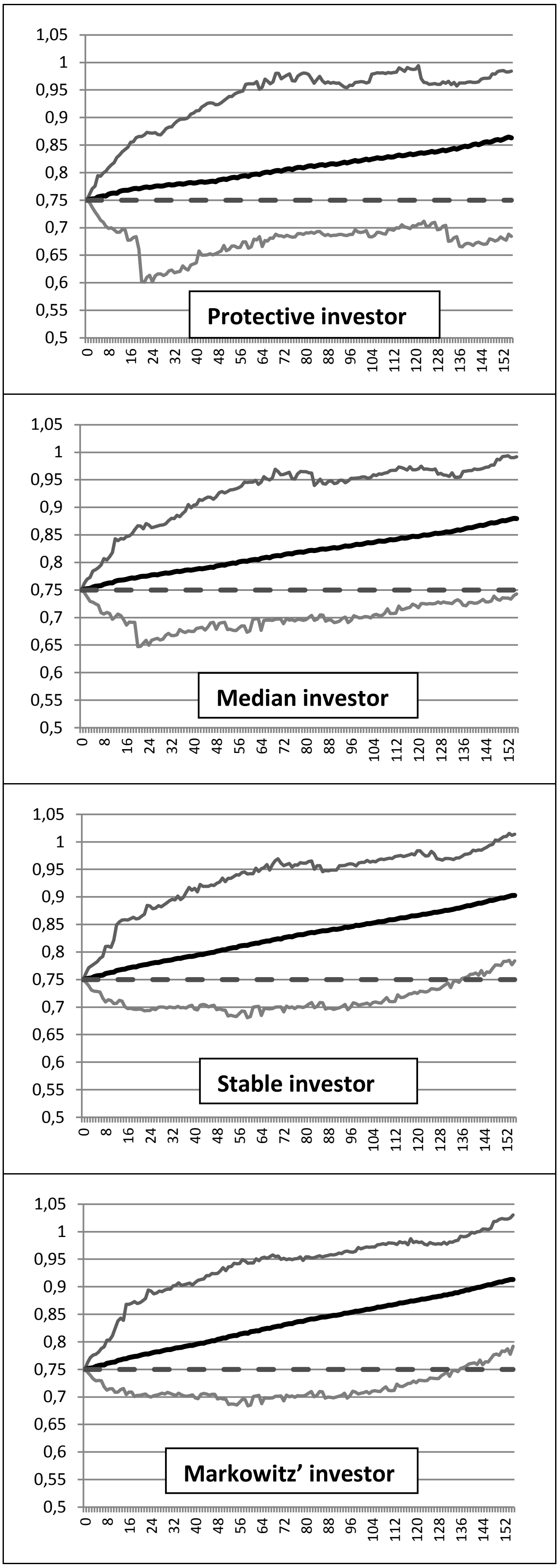
[6] The trends and conclusions are the same for the defensive and aggressive portfolios.

[7] The average increase of all the ERAs, is mainly due to the increase in the risk of the bond index, relative to all equity indices.

Exhibit 7 - Evolution of the portfolio allocation of an average risk portfolio (ERA = 75%) for 4 different investors. The portfolios are reallocated every 4 weeks on the basis of the risk profile over the last 18 months.



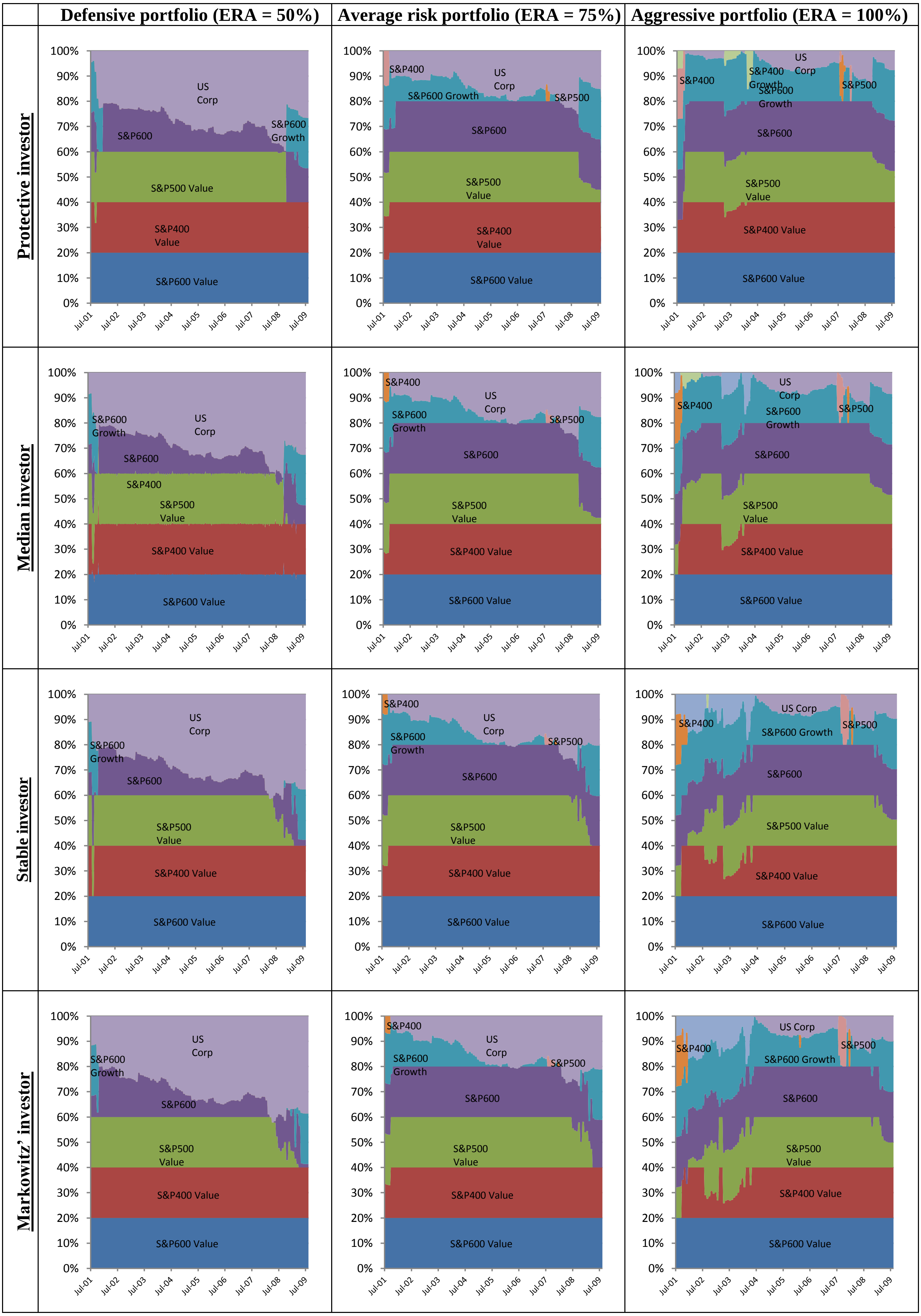
Furthermore, the surface above the dashed line is smaller for the Bell approaches than for the variance. The 5% quantile, providing the lowest value of the ERA under a level of confidence of 95%, shows it is more likely to remain under the target value with a Bell's risk optimization than with the Markowitz approach. Even though the overall portfolio risk increases over time, the surface below the dashed line is larger for the Bell's risk measure approaches than for the Markowitz one.

Time consistency with rebalancing

Although interesting to observe and informative, the previous exercise is not very close to the reality of asset managers. Now, we do not consider the different allocations of each investor separately, but we study the dynamics of the 12 portfolios reallocated every four weeks. Exhibit 8 displays the evolution of the weights of the 10 indices for the defensive, average risk and aggressive portfolios constructed for the 4 different perceptions of risk considered.

For the defensive portfolios, the evolution of the weights on October 2008 are rather revealing about the rapidity of reaction of the risk measure to a change in the risk profile of the assets. Indeed, the $\text{Bell}_{C=34}$ optimization rejects the S&P500 Value index in once, whereas it takes until April 2009, for the Markowitz model to reject it. Indeed, if we look closely to the S&P500 Value index, the volatility of its returns started to increase in early 2008, and the index suffered highly negative returns on September 26th, 2008 and until October 10th. After that date, successive highly positive and negative returns follow one another until the last day of the database. As the portfolio is rebalanced on September 19th, and on October 17th, it takes one re-allocation for the Stable investor to run away from the source of extreme losses, whereas the Markowitz investor have already started to move away, but gradually, from the increasing risk index in April 2008. This example illustrates the importance of taking into account higher moments of the distribution of returns.

Exhibit 8 - Evolution of the allocations of portfolios reallocated every four weeks over the last 18 months.



It justifies the significance of the risk perception in the portfolio allocation process. The other indices driving the risk profiles of the portfolios are the Bond index, and the S&P 600 Growth index for all portfolios, the S&P 600 for the defensive ones, and the Growth indices as well as the S&P500 and 400 for the aggressive portfolios. The main index driving the return is the S&P 600 Value index. The other "return drivers" are the S&P400 Value for defensive and average-risk portfolios, and the S&P600 for the average-risk and aggressive portfolios.

Next, we study the evolution of the risk taken by the investors through time. Exhibit 9, 10 and 11 compare the ERA of the actively managed portfolios for the 4 investors. They report the average ERA on the day of the allocation (ERA$_t$) and the average ERA four weeks later, just before the next allocation (ERA$_{t+4}$). The average differences between both ERA (bias) and the Root Mean Square Errors (RMSE), expressed in percentage of the ERA$_t$, are also reported. The shaded sections highlight the measures when the right method is used for the right investor. Exhibit 9 (resp. 10 and 11) displays those results for the defensive (resp. average-risk and aggressive) portfolios. The portfolios of the protective investors tend to be perceived as riskier than expected by the other investors, whereas the portfolios of the Markowitz and stable investors look less risky than expected for the protector and median investors. This might be due to an overall increase in the "kurtosis" type of risk of the market comparatively to the variance, as already suggested in previous sub-section.

The bias show that the methodology performs well in regards to its objective, as all bias are lower than 1%, except for the Markowitz optimization where it nonetheless does not outreach the 1.4%. However, if we look at the RSME, we clearly see that, if one does not know the profile of your investor, the Markowitz risk measure is most likely to be misleading. It can produce ERA more than 20% far away from its objective. This even worsens for higher degree of risk. For the Bell Risk Measure, even if one mistakenly assesses the perception parameter, the highest RMSE is half the Markowitz', and the method even improves comparatively when the level of risk rises, with a RMSE of maximum 6.7% for an ERA of 100%.

Exhibit9

Equivalent Risky Allocation for defensive investors.

	Investors with risk aversion such as ERA = 50%			
	Protector	Median	Stable	Markowitz
Panel A - Optimization with Markowitz				
ERAt	48.3%	49.6%	50.1%	**50.0%**
ERA$_{t+4}$	48.9%	50.2%	50.7%	**50.7%**
bias	1.2%	1.2%	1.4%	**1.4%**
RMSE	21.8%	13.5%	7.4%	**6.5%**
Panel B - Optimization of Bell Utility Function with C = 5				
ERAt	45.6%	48.5%	**50.0%**	50.3%
ERA$_{t+4}$	46.0%	48.9%	**50.4%**	50.7%
bias	0.8%	0.8%	**0.9%**	0.9%
RMSE	10.5%	5.8%	**3.5%**	3.5%
Panel C - Optimization of Bell Utility Function with C = 17				
ERAt	47.1%	**50.0%**	51.6%	51.8%
ERA$_{t+4}$	47.6%	**50.4%**	52.0%	52.3%
bias	0.9%	**0.9%**	0.9%	0.9%
RMSE	6.9%	**4.2%**	5.5%	6.3%
Panel D - Optimization of Bell Utility Function with C = 34				
ERAt	**50.0%**	52.8%	54.4%	54.7%
ERA$_{t+4}$	**50.5%**	53.3%	54.9%	55.1%
bias	**1.0%**	0.9%	0.9%	0.9%
RMSE	**6.1%**	6.6%	9.1%	10.0%

Exhibit10

Equivalent Risky Allocation for average-risk investors.

	Investor with risk aversion such as ERA = 75%			
	Protector	Median	Stable	Markowitz
Panel A - Optimization with Markowitz				
ERAt	76.7%	76.1%	75.4%	**75.0%**
ERAt+4	77.6%	77.0%	76.3%	**76.0%**
Bias	1.2%	1.2%	1.3%	**1.3%**
RMSE	22.9%	13.8%	7.0%	**5.9%**
Panel B - Optimization of Bell Utility Function with C = 5				
ERAt	72.8%	74.3%	**75.0%**	75.0%
ERAt+4	73.4%	74.9%	**75.5%**	75.5%
Bias	0.8%	0.8%	**0.7%**	0.7%
RMSE	9.1%	5.2%	**2.9%**	2.9%
Panel C - Optimization of Bell Utility Function with C = 17				
ERAt	73.5%	**75.0%**	75.7%	75.8%
ERAt+4	74.0%	**75.5%**	76.2%	76.3%
Bias	0.8%	**0.7%**	0.7%	0.7%
RMSE	5.8%	**3.7%**	4.4%	5.3%
Panel D - Optimization of Bell Utility Function with C = 34				
ERAt	**75.0%**	76.4%	77.1%	77.2%
ERAt+4	**75.6%**	77.0%	77.6%	77.7%
Bias	**0.7%**	0.7%	0.7%	0.7%
RMSE	**4.5%**	4.7%	6.7%	7.6%

Exhibit11

Equivalent Risky Allocation for aggressive investors.

	Investor with risk aversion such as ERA = 100%			
	Protector	Median	Stable	Markowitz
Panel A - Optimization with Markowitz				
ERAt	107.5%	103.5%	100.8%	**100.0%**
ERAt+4	108.6%	104.5%	101.9%	**101.1%**
Bias	1.0%	1.0%	1.1%	**1.1%**
RMSE	23.9%	14.1%	6.6%	**5.5%**
Panel B - Optimization of Bell Utility Function with C = 5				
ERAt	101.2%	100.5%	**100.0%**	99.8%
ERAt+4	101.5%	100.8%	**100.4%**	100.2%
Bias	0.3%	0.3%	**0.4%**	0.4%
RMSE	5.6%	3.3%	**1.9%**	2.0%
Panel C - Optimization of Bell Utility Function with C = 17				
ERAt	101.1%	**100.0%**	99.2%	98.9%
ERAt+4	101.5%	**100.4%**	99.6%	99.3%
Bias	0.4%	**0.4%**	0.4%	0.4%
RMSE	4.2%	**2.6%**	3.6%	4.5%
Panel D - Optimization of Bell Utility Function with C = 34				
ERAt	**100.0%**	99.0%	98.3%	98.0%
ERAt+4	**100.4%**	99.5%	98.7%	98.5%
Bias	**0.4%**	0.4%	0.5%	0.5%
RMSE	**3.3%**	3.8%	5.7%	6.7%

Portfolio Consistency

Now that we have checked for the time consistency of the risk measure, one of the most important things for the investor is to check afterwards if the portfolio he held during the whole period of investment has realized its objectives in term of risk.

To test the portfolio consistency of our 12 portfolios, we rebuild the historical of the rebalanced portfolios and compute their ERA on the last day of the holding period, that is, on August 14th, 2009. Exhibit 14 displays the final ERAs of the 12 portfolios, constructed on June 29th, 2001, and rebalanced every four weeks until July 24th, 2009. The last column displays the maximum spread between the ERA of the investor for whom the risk measure used for the optimization is the right one (in bold), and the ERA of the others investors.

Results in Exhibit 12 show that the portfolios established for the different investors (in bold) have been riskier than expected. Whereas the portfolios were reallocated every four weeks, with the constraint that the ERA must be lower than 50%, 75% and 100%, the final ERA are higher than the target values. This is partially due to an increase of the global risk of the "return drivers". Indeed, the variance (resp. kurtosis) of the S&P600 Value index has, for instance, increased from 50% to 184% (resp. from 18% to 215%) of the benchmark variance (resp. kurtosis). Nonetheless, we can observe that the relative increase of the ERA is lower the higher the expected ERA. This might be due to the S&P600 index and, to a smaller extent, to S&P 500 and 400 indices, which have experienced a reduction in variance and kurtosis over the tested period.

Exhibit12

Final Equivalent Risky Allocation of 12 rebalanced portfolios for 4 different investors (Protector, Median, Stable, Markowitz).

Risk Measure	ERA	Investors				Max spread
		Protective	Median	Stable	Markowitz	
Variance	Defensive	0.45	0.51	0.54	**0.55**	0.10
	Average	0.72	0.77	0.79	**0.80**	0.08
	Aggressive	0.99	1.01	1.03	**1.03**	0.04
Bell_5	Defensive	0.43	0.49	**0.54**	0.55	0.11
	Average	0.73	0.77	**0.80**	0.81	0.07
	Aggressive	1.00	1.02	**1.03**	1.04	0.03
Bell_17	Defensive	0.48	**0.54**	0.58	0.59	0.06
	Average	0.77	**0.80**	0.83	0.83	0.03
	Aggressive	1.04	**1.04**	1.05	1.05	0.01
Bell_34	Defensive	**0.56**	0.61	0.64	0.65	0.09
	Average	**0.81**	0.84	0.86	0.87	0.06
	Aggressive	**1.05**	1.05	1.05	1.06	0.01

The spread between the ERAs of the different investors for a same portfolio illustrates the consequence of a wrong profiling. A misunderstanding of the investor's perception of risk could make her endure a risk up to 11% away from his objective. Besides, using the wrong risk measure for an investor appears to have more impact for defensive portfolios, as the maximum spread is higher for those portfolios.

Conclusion

This study has demonstrated the relevance of the perception of risk, defined as the subjective judgment of an investor about the characteristics and severity of a risk, for portfolio allocation. Although confirmed several times in experimental finance (see Cooper et al. [1988]

and Sitkin and Weingart [1995] among others), none of the authors were able to reconcile their perception driven risk measure with the theory of utility function until Bell (Weber and Milliman [1997]). Bell derives a risk measure from a linex utility function and approximates the risk perception of the investor through an individualized specific parameter.

The Bell Risk Measure allows us to adequately address the request of an increasing number of authors advocating for the introduction of higher moments of the distribution of returns when optimizing portfolios. Unfortunately, the Bell Risk Measure is much more complicated to interpret than the mere variance. We develop therefore a standardization technique that simplifies its interpretation but keep its individualization specificity: the equivalent risky allocation (ERA). This measure simply expresses the risk measure of an asset in terms of a percentage of the wealth to invest in a selected benchmark to obtain the same risk value.

Through an exercise of portfolio optimization using nine equity indices and a bond index, we test the time and portfolio consistency of the measure and compare it to the variance of Markowitz, which we also express in terms of ERA. Although the variance is a particular case of the Bell Risk Measure with C that tends to zero, our research shows the relevance of the recognition of the risk perception in portfolio allocation. Moreover, we demonstrate the consequences of a misspecification of the risk profiles of the investors, who are brought to encounter a too high risk or to miss potentially higher returns.

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